

INFINITELY MANY POSITIVE SOLUTIONS FOR NONLINEAR FRACTIONAL SCHRÖDINGER EQUATIONS

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ABSTRACT. We consider the following nonlinear fractional Schrödinger equation

$$(-\Delta)^s u + u = K(|x|)u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N,$$

where $K(|x|)$ is a positive radial function, $N \geq 2$, $0 < s < 1$, $1 < p < \frac{N+2s}{N-2s}$. Under some asymptotic assumptions on $K(x)$ at infinity, we show that this problem has infinitely many non-radial positive solutions, whose energy can be made arbitrarily large.

Key words : fractional Laplacian; nonlinear Schrödinger equation; reduction method.

AMS Subject Classifications: 35J20, 35J60

1. INTRODUCTION

In this paper, we consider the following nonlinear fractional Schrödinger equation

$$(-\Delta)^s u + u = K(x)u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N, \quad u \in H^s(\mathbb{R}^N) \quad (1.1)$$

with dimension $N \geq 2$, where $K(x)$ is a positive continuous potential, $0 < s < 1$, $1 < p < 2_*(s) - 1$, $2_*(s) = \frac{2N}{N-2s}$ and

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}.$$

The fractional Laplacian of a function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is expressed by the formular

$$(-\Delta)^s f(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x - y|^{N+2s}} dy = C_{N,s} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{f(x) - f(y)}{|x - y|^{N+2s}} dy, \quad (1.2)$$

where $C_{N,s}$ is some normalization constant (See Sect. 2).

Problem (1.1) arises from looking for standing waves $\Psi(t, x) = \exp(iEt)u(x)$ for the following nonlinear Schrödinger equations

$$i \frac{\partial \Psi}{\partial t} = (-\Delta)^s \Psi + (1 + E)\Psi - K(x)|\Psi|^{p-1}\Psi, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \quad (1.3)$$

where i is the imaginary unit. This equation is of particular interest in fractional quantum mechanics for the study of particles on stochastic fields modelled by Lévy processes. A path integral over the Lévy flights paths and a fractional Schrödinger equation of fractional quantum mechanics are formulated by Laskin [20] from the idea of Feynman and Hibbs's path integrals (see also [21]).

The Lévy processes occur widely in physics, chemistry and biology. The stable Lévy processes that give rise to equations with the fractional Laplacians have recently attracted

much research interest, and there are a lot of results in the literature on the existence of such solutions, e.g., [1, 3–5, 8, 9, 14, 19, 22, 27, 31] and the references therein.

A partner problem of (1.1) is the following scalar field equation

$$(-\Delta)^s u + V(x)u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^N, \quad u \in H^s(\mathbb{R}^N). \quad (1.4)$$

In the sequel, we will assume that V, K is bounded, and $V(x) \geq V_0 > 0$, $K(x) \geq K_0 > 0$.

It is known, but not completely trivial, that $(-\Delta)^s$ reduces to the standard Laplacian $-\Delta$ as $s \rightarrow 1$. When $s = 1$, the classical nonlinear Schrödinger equation has been extensively studied in the last thirty years. If

$$\inf_{x \in \mathbb{R}^N} V(x) < \lim_{|x| \rightarrow \infty} V(x), \quad (\text{or } \sup_{x \in \mathbb{R}^N} K(x) > \lim_{|x| \rightarrow \infty} K(x)), \quad (1.5)$$

then, using the concentration compactness principle [23, 24], one can show that (1.1) and (1.4) has a least energy solution. See for example [13, 23, 24]. But if (1.5) does not hold, (1.1) or (1.4) may not have a least energy solution. So, in this case, one naturally needs to find solutions with higher energy. Recently, Cerami et al. [6] showed that problem (1.4) with $s = 1$ has infinitely many sign-changing solutions if $V(x)$ goes to its limit at infinity from below at a suitable rate. In [29], Wei and Yan gave a surprising result which says that (1.1) or (1.4) with $s = 1$ and $V(x)$ or $K(x)$ being radial has solutions with large number of bumps near infinity and the energy of this solutions can be very large. This kind of results was generalized by Ao and Wei in [2] very recently to the case in which $V(x)$ or $K(x)$ does not have any symmetry assumption. For more results on (1.1) and (1.4) with $s = 1$, we can refer to [6, 7, 12, 13] and the references therein.

When $0 < s < 1$, Chen and Zheng [10] studied the following singularly perturbed problem

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^N. \quad (1.6)$$

They showed that when $N = 1, 2, 3$, ε is sufficiently small, $\max\{\frac{1}{2}, \frac{n}{4}\} < s < 1$ and V satisfies some smoothness and boundedness assumptions, equation (1.6) has a nontrivial solution u_ε concentrated to some single point as $\varepsilon \rightarrow 0$. Very recently, in [11], Dávila, del Pino and Wei generalized various existence results known for (1.6) with $s = 1$ to the case of fractional Laplacian. For results which are not for singularly perturbed type of (1.1) and (1.4) with $0 < s < 1$, the readers can refer to [3, 14, 26, 28] and the references therein.

As far as we know, it seems that there is no result on the existence of multiple solutions of equation (1.1) which is not a singularly perturbed problem. The aim of this paper is to obtain infinitely many non-radial positive solutions for (1.1) whose functional energy are very large, under some assumptions for $K(x) = K(|x|) > 0$ near the infinity.

Let

$$\frac{N + 2s}{N + 2s + 1} < m < N + 2s. \quad (1.7)$$

We assume that $0 < K(|x|) \in C(\mathbb{R}^N)$ satisfies the following condition at infinity

$$(K) : \quad K(r) = K_0 - \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right), \quad \text{as } r \rightarrow +\infty,$$

for some $a > 0$, $\theta > 0$. Without loss of generality, we may assume that $K_0 = 1$.

Our main result in this paper can be stated as follows

Theorem 1.1. *Suppose that $N \geq 2$, $0 < s < 1$, $1 < p < 2_*(s) - 1$. If $K(r)$ satisfies (K), then problem (1.1) has infinitely many non-radial positive solutions.*

Remark 1.2. The radial symmetry can be replaced by the following weaker symmetry assumption: after suitably rotating the coordinated system,

- (i) $K(x) = K(x', x'') = K(|x'|, |x_3|, \dots, |x_N|)$, where $x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$,
- (ii) $K(x) = K_0 - \frac{a}{|x|^m} + O(\frac{1}{|x|^{m+\theta}})$ as $|x| \rightarrow +\infty$, where $a > 0$, $\theta > 0$ and $K_0 > 0$ are some constants.

Remark 1.3. Using the same argument, we can prove that if

$$V(x) = V(x', x'') = V(|x'|, |x_3|, \dots, |x_N|) = V_0 + \frac{a}{|x|^m} + O(\frac{1}{|x|^{m+\theta}}), \text{ as } |x| \rightarrow \infty,$$

for some constants $V_0 > 0$, $a > 0$, and $\theta > 0$. Then problem (1.4) has infinitely many positive non-radial solutions.

Before we close this introduction, let us outline the main idea in the proof of Theorem 1.1.

We will use the unique ground state U of

$$(-\Delta)^s u + u = u^p, \quad u > 0, \quad x \in \mathbb{R}^N, \quad u(0) = \max_{\mathbb{R}^N} u(x) \quad (1.8)$$

to build up the approximate solutions for (1.1). It is well known that when $s = 1$, the ground state solution of (1.8) decays exponentially at infinity. But from [15, 16], we see that when $s \in (0, 1)$, the unique ground solution of (1.8) decays like $\frac{1}{|x|^{N+2s}}$ when $|x| \rightarrow \infty$.

Let any integer $k > 0$, define

$$x^i = \left(r \cos \frac{2(i-1)\pi}{k}, r \sin \frac{2(i-1)\pi}{k}, 0 \right), \quad i = 1, \dots, k,$$

where 0 is the zero vector in \mathbb{R}^{N-2} , $r \in [r_0 k^{\frac{N+2s}{N+2s-m}}, r_1 k^{\frac{N+2s}{N+2s-m}}]$ for some $r_1 > r_0 > 0$.

Set $x = (x', x'')$, $x' \in \mathbb{R}^2$, $x'' \in \mathbb{R}^{N-2}$. Define

$$\mathcal{H} = \left\{ u : u \in H^s(\mathbb{R}^N), u \text{ is even in } x_i, i = 2, \dots, N, \right.$$

$$\left. u(r \cos \theta, r \sin \theta, x'') = u(r \cos(\theta + \frac{2j\pi}{k}), r \sin(\theta + \frac{2j\pi}{k}), x''), j = 1, \dots, k-1 \right\}.$$

Write

$$U_r(x) = \sum_{i=1}^k U_{x^i}(x),$$

where $U_{x^i}(x) = U(x - x^i)$.

We will prove Theorem 1.1 by proving the following result

Theorem 1.4. *Under the assumption of Theorem (1.1), there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, (1.1) has a solution u_k of the form*

$$u_k = U_r(x) + \omega_r$$

where $\omega_r \in \mathcal{H}$, $r \in [r_0 k^{\frac{N+2s}{N+2s-m}}, r_1 k^{\frac{N+2s}{N+2s-m}}]$ and as $k \rightarrow +\infty$,

$$\int_{\mathbb{R}^{2N}} \frac{|\omega_r(x) - \omega_r(y)|^2}{|x - y|^{N+2s}} + \int_{\mathbb{R}^N} \omega_r^2 \rightarrow 0.$$

The idea of our proof is inspired by that of [29] where infinitely many positive non-radial solutions to a nonlinear Schrödinger equations (1.3) with $s = 1$ are obtained when the potential approaches to a positive constant algebraically at infinity. We will use the well-known Lyapunov-Schmidt reduction scheme to transfer our problem to a maximization problem of a one-dimensional function in a suitable range. Compared with the operator $-\Delta$, which is local, the operator $(-\Delta)^s$ with $0 < s < 1$ on \mathbb{R}^N is nonlocal. So it is expected that the standard techniques for $-\Delta$ do not work directly. In particular, when we try to find spike solutions for (1.1) with $0 < s < 1$, $(-\Delta)^s$ may kill bumps by averaging on the whole \mathbb{R}^N . For example, the ground state for (1.1) with $0 < s < 1$ decays algebraically at infinity, which is a contrast to the fact that the ground state for $-\Delta$ decays exponentially at infinity. This kind of property requires us to establish some new basic estimates and give a precise estimate on the energy of the approximate solutions.

This paper is organized as follows. In Sect.2, we will give some preliminary properties related to the fractional Laplacian operator. In Sect.3, we will establish some preliminary estimate. We will carry out a reduction procedure and study the reduced one dimensional problem to prove Theorems 1.4 in Sect.4. In Appendix, some basic estimates and an energy expansion for the functional corresponding to problem (1.1) will be established.

2. BASIC THEORY ON FRACTIONAL LAPLACIAN OPERATOR

In this section, we recall some properties of the fractional order Sobolev space and the ground state solution U of the limit equation (1.8).

Let $0 < s < 1$. Various definitions of the fractional Laplacian $(-\Delta)^s f(x)$ of a function f defined in \mathbb{R}^N are available, depending on its regularity and growth properties.

It can be defined as a pseudo-differential operator

$$\widehat{(-\Delta)^s f}(\xi) = |\xi|^{2s} \widehat{f}(\xi),$$

where $\widehat{\cdot}$ is Fourier transform. When f have some sufficiently regular, the fractional Laplacian of a function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is expressed by the formular

$$(-\Delta)^s f(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x - y|^{N+2s}} dy = C_{N,s} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{f(x) - f(y)}{|x - y|^{N+2s}} dy,$$

where $C_{N,s} = \pi^{-(2s+N/2)} \frac{\Gamma(N/2+s)}{\Gamma(-s)}$. This integral makes sense directly when $s < \frac{1}{2}$ and $f \in C^{0,\alpha}(\mathbb{R}^N)$ with $\alpha > 2s$, or if $f \in C^{1,\alpha}(\mathbb{R}^N)$, $1 + 2\alpha > 2s$. It is well known that $(-\Delta)^s$ on \mathbb{R}^N with $0 < s < 1$ is a nonlocal operator. In the remarkable work of Caffarelli and Silvestre [4], this nonlocal operator was expressed as a generalized Dirichlet-Neumann map for a certain elliptic boundary value problem with nonlocal differential operator defined on the upper half-space $\mathbb{R}_+^{N+1} := \{(x, y) : x \in \mathbb{R}^N, y > 0\}$. That is, for a function

$f : \mathbb{R}^N \rightarrow \mathbb{R}$, we consider the extension $u(x, y) : \mathbb{R}^N \times [0, +\infty) \rightarrow \mathbb{R}$ that satisfies the equation

$$u(x, 0) = f(x) \quad (2.1)$$

$$\Delta_x u + \frac{1-2s}{y} u_y + u_{yy} = 0. \quad (2.2)$$

The equation (2.2) can also be written as

$$\operatorname{div}(y^{1-2s} \nabla u) = 0, \quad (2.3)$$

which is clearly the Euler-Lagrange equation for the functional

$$J(u) = \int_{y>0} |\nabla u|^2 y^{1-2s} dx dy.$$

Then, it follows from [4] that

$$C(-\Delta)^s f = \lim_{y \rightarrow 0^+} -y^{1-2s} u_y = \frac{1}{2s} \lim_{y \rightarrow 0^+} \frac{u(x, y) - u(x, 0)}{y^{2s}}.$$

When $s \in (0, 1)$, the space $H^s(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$ is defined by

$$\begin{aligned} H^s(\mathbb{R}^N) &= \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\} \\ &= \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2s}) |\widehat{u}(\xi)|^2 d\xi < +\infty \right\} \end{aligned}$$

and the norm is

$$\|u\|_s := \|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{1}{2}}.$$

Here the term

$$[u]_s := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}$$

is the so-called Gagliardo (semi) norm of u . The following identity yields the relation between the fractional operator $(-\Delta)^s$ and the fractional Laplacian Sobolev space $H^s(\mathbb{R}^N)$,

$$[u]_{H^s(\mathbb{R}^N)} = C \left(\int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} = C \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}$$

for a suitable positive constant C depending only s and N .

Clearly, $\|\cdot\|_{H^s(\mathbb{R}^N)}$ is a Hilbertian norm induced by the inner product

$$\begin{aligned} \langle u, v \rangle_{H^s(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} u(x)v(x) dx \\ &= \langle u, v \rangle_s + \langle u, v \rangle_{L^2(\mathbb{R}^N)}. \end{aligned}$$

On the Sobolev inequality and the compactness of embedding, one has

Theorem 2.1. [25] *The following imbeddings are continuous:*

- (1) $H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, $2 \leq q \leq \frac{2N}{N-2s}$, if $N > 2s$,
- (2) $H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, $2 \leq q \leq +\infty$, if $N = 2s$,
- (3) $H^s(\mathbb{R}^N) \hookrightarrow C_b^j(\mathbb{R}^N)$, if $N < 2(s-j)$.

Moreover, for any $R > 0$ and $p \in [1, 2_*(s))$ the embedding $H^s(B_R) \hookrightarrow L^p(B_R)$ is compact, where

$$C_b^j(\mathbb{R}^N) = \left\{ u \in C^j(\mathbb{R}^N) : D^K u \text{ is bounded on } \mathbb{R}^N \text{ for } |K| \leq j \right\}.$$

Now, we recall some known results for the limit equation (1.8). If $s = 1$, the uniqueness and non-degeneracy of the ground state U for (1.8) is due to [18]. In the celebrated paper [15], Frank and Lenzemann proved the uniqueness of ground state solution $U(x) = U(|x|) \geq 0$ for $N = 1, 0 < s < 1, 1 < p < 2_*(s) - 1$. Very recently, Frank, Lenzemann and Silvestre [16] obtained the non-degeneracy of ground state solutions for (1.8) in arbitrary dimension $N \geq 1$ and any admissible exponent $1 < p < 2_*(s) - 1$.

For convenience, we summarize the properties of the ground state U of (1.8) which can be found in [15, 16].

Theorem 2.2. *Let $N \geq 1$, $s \in (0, 1)$ and $1 < p < 2_*(s) - 1$. Then the following hold.*

- (i) (Uniqueness) *The ground state solution $U \in H^s(\mathbb{R}^N)$ for equation (1.8) is unique.*
- (ii) (Symmetry, regularity and decay) *$U(x)$ is radial, positive and strictly decreasing in $|x|$. Moreover, the function U belongs to $H^{2s+1}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ and satisfies*

$$\frac{C_1}{1 + |x|^{N+2s}} \leq U(x) \leq \frac{C_2}{1 + |x|^{N+2s}} \quad \text{for } x \in \mathbb{R}^N,$$

with some constants $C_2 \geq C_1 > 0$.

- (iii) (Non-degeneracy) *The linearized operator $L_0 = (-\Delta)^{s+1-p}|U|^{p-1}$ is non-degenerate, i.e., its kernel is given by*

$$\ker L_0 = \text{span}\{\partial_{x_1} U, \partial_{x_2} U, \dots, \partial_{x_N} U\}.$$

By Lemma C.2 of [16], it holds that, for $j = 1, \dots, N$, $\partial_{x_j} U$ has the following decay estimate,

$$|\partial_{x_j} U| \leq \frac{C}{1 + |x|^{N+2s}}.$$

From Theorem 2.2, the ground bound state solution like $\frac{1}{|x|^{N+2s}}$ when $|x| \rightarrow +\infty$. Fortunately, this polynomial decay is enough for us in the estimates of our proof.

3. SOME PRELIMINARIES

Let

$$Z_i = \frac{\partial U_{x^i}}{\partial r}, \quad i = 1, \dots, k,$$

where $x^i = (r \cos \frac{2(i-1)\pi}{k}, r \sin \frac{2(i-1)\pi}{k}, 0)$ and

$$r \in [r_0 k^{\frac{N+2s}{N+2s-m}}, r_1 k^{\frac{N+2s}{N+2s-m}}]$$

for some $r_1 > r_0 > 0$.

Define

$$E = \left\{ v \in \mathcal{H} : \sum_{i=1}^k \int_{\mathbb{R}^N} U_{x^i}^{p-1} Z_i v = 0 \right\}.$$

The norm of $H^s(\mathbb{R}^N)$ is defined by:

$$\|v\|_s = \sqrt{\langle v, v \rangle}, \quad v \in H^s(\mathbb{R}^N),$$

where

$$\begin{aligned} \langle v_1, v_2 \rangle &= \langle v_1, v_2 \rangle_s + \int_{\mathbb{R}^N} v_1 v_2 \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} u(x) v(x) dx. \end{aligned}$$

The variational functional corresponding to (1.1) is

$$I(u) = \frac{1}{2} \langle u, u \rangle_s + \frac{1}{2} \int_{\mathbb{R}^N} u^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} K(x) |u|^{p+1}.$$

Let

$$J(\varphi) = I(U_r + \varphi) = I\left(\sum_{i=1}^k U_{x^i} + \varphi\right), \quad \varphi \in E.$$

Expand $J(\varphi)$ as follows:

$$J(\varphi) = J(0) + l(\varphi) + \frac{1}{2} \langle L(\varphi), \varphi \rangle + R(\varphi), \quad \varphi \in E, \quad (3.1)$$

where

$$\begin{aligned} l(\varphi) &= \int_{\mathbb{R}^N} \sum_{i=1}^k U_{x^i}^p \varphi - \int_{\mathbb{R}^N} K(x) \left(\sum_{i=1}^k U_{x^i} \right)^p \varphi, \\ \langle L(\varphi), \varphi \rangle &= \langle \varphi, \varphi \rangle_s + \int_{\mathbb{R}^N} (\varphi^2 - pK(x) U_r^{p-1} \varphi^2) \end{aligned}$$

and

$$R(\varphi) = -\frac{1}{p+1} \int_{\mathbb{R}^N} K(x) \left((U_r + \varphi)^{p+1} - U_r^{p+1} - (p+1) U_r^p \varphi - \frac{1}{2} (p+1) p U_r^{p-1} \varphi^2 \right).$$

In order to find a critical point for $J(\varphi)$, we need to discuss each term in the expansion (3.1).

Lemma 3.1. *There is a constant $C > 0$ independent of k such that*

$$\|R'(\varphi)\| \leq C \|\varphi\|_s^{\min\{p, 2\}},$$

$$\|R''(\varphi)\| \leq C \|\varphi\|_s^{\min\{p-1, 1\}}.$$

Proof. By direct calculation, we know that

$$\begin{aligned}\langle R'(\varphi), \psi \rangle &= - \int_{\mathbb{R}^N} K(x) \left((U_r + \varphi)^p - U_r^p - pU_r^{p-1}\varphi \right) \psi, \\ \langle R''(\varphi)(\psi, \xi) \rangle &= -p \int_{\mathbb{R}^N} K(x) \left((U_r + \varphi)^{p-1} - U_r^{p-1} \right) \psi \xi.\end{aligned}$$

Firstly, we deal with the case $p > 2$. Since

$$|\langle R'(\varphi), \psi \rangle| \leq C \int_{\mathbb{R}^N} U_r^{p-2} |\varphi|^2 |\psi| \leq C \left(\int_{\mathbb{R}^n} (U_r^{p-2} |\varphi|^2)^{\frac{p+1}{p}} \right)^{\frac{p}{p+1}} \|\psi\|_s,$$

we find

$$\|R'(\varphi)\| \leq C \left(\int_{\mathbb{R}^n} (U_r^{p-2} |\varphi|^2)^{\frac{p+1}{p}} \right)^{\frac{p}{p+1}}.$$

On the other hand, it follows from Lemma A.2 that U_r is bounded. Since $2 < \frac{2(p+1)}{p} < p+1$, we obtain

$$\|R'(\varphi)\| \leq C \left(\int_{\mathbb{R}^n} |\varphi|^{\frac{2(p+1)}{p}} \right)^{\frac{p}{p+1}} \leq C \|\varphi\|_s^2.$$

For the estimate of $\|R''(\varphi)\|$, we have

$$\begin{aligned} |R''(\varphi)(\psi, \xi)| &\leq C \int_{\mathbb{R}^N} U_r^{p-2} |\varphi| |\psi| |\xi| \\ &\leq C \int_{\mathbb{R}^N} |\varphi| |\psi| |\xi| \leq C \left(\int_{\mathbb{R}^N} |\varphi|^3 \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^N} |\psi|^3 \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^N} |\xi|^3 \right)^{\frac{1}{3}} \\ &\leq C \|\varphi\|_s \|\psi\|_s \|\xi\|_s. \end{aligned}$$

So,

$$\|R''(\varphi)\| \leq C \|\varphi\|_s.$$

Using the same argument, for the case $1 < p \leq 2$, we also can obtain that

$$\begin{aligned}\langle R'(\varphi), \psi \rangle &\leq C \int_{\mathbb{R}^N} \varphi^p \psi \leq C \|\varphi\|_s^p \|\psi\|_s, \\ \langle R'(\varphi)(\psi, \xi) \rangle &\leq C \int_{\mathbb{R}^N} \varphi^{p-1} \psi \xi \leq C \|\varphi\|_s^{p-1} \|\psi\|_s \|\xi\|_s.\end{aligned}$$

□

4. THE FINITE-DIMENSIONAL REDUCTION AND PROOF OF THE MAIN RESULTS

In this section, we intend to prove the main theorem by the Lyapunov-Schmidt reduction.

Associated to the quadratic form $L(\varphi)$, we define L to be a bounded linear map from E to E , such that

$$\langle Lv_1, v_2 \rangle = \langle v_1, v_2 \rangle_s + \int_{\mathbb{R}^N} \left(v_1 v_2 - pK(x) U_r^{p-1} v_1 v_2 \right), \quad v_1, v_2 \in E.$$

In this paper, we always assume

$$r \in S_k := \left[\left(\left(\frac{B_0(N+2s)}{B_1 m} - \alpha \right)^{\frac{1}{N+2s-m}} k^{\frac{N+2s}{N+2s-m}}, \left(\left(\frac{B_0(N+2s)}{B_1 m} + \alpha \right)^{\frac{1}{N+2s-m}} k^{\frac{N+2s}{N+2s-m}} \right), \quad (4.1)$$

where $\alpha > 0$ is a small constant, B_0 and B_1 are defined in Proposition A.3.

Next, we show the invertibility of L in E .

Proposition 4.1. *There exists an integer $k_0 > 0$, such that for $k \geq k_0$, there is a constant $\rho > 0$ independent of k , satisfying that for any $r \in S_k$,*

$$\|Lu\| \geq \rho \|u\|_s, \quad u \in E.$$

Proof. We argue by contradiction. Suppose that there are $n \rightarrow +\infty$, $r_k \in S_k$, and $u_n \in E$, such that

$$\|Lu_n\| = o(1) \|u_n\|_s, \quad \|u_n\|_s^2 = k.$$

Recall

$$\Omega_i = \left\{ x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\langle \frac{x'}{|x'|}, \frac{x^i}{|x^i|} \right\rangle \geq \cos \frac{\pi}{k} \right\}, \quad i = 1, 2, \dots, k.$$

By symmetry, we have

$$\begin{aligned} & \int_{\Omega_1} \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} + \int_{\Omega_1} (u_n \varphi - pK(x)U_{r_k}^{p-1}u_n \varphi) \\ &= \frac{1}{k} \langle Lu_n, \varphi \rangle = o\left(\frac{1}{\sqrt{k}}\right) \|\varphi\|_s, \quad \varphi \in E. \end{aligned} \quad (4.2)$$

In particular,

$$\int_{\Omega_1} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} + \int_{\Omega_1} (|u_n|^2 - pK(x)U_{r_k}^{p-1}|u_n|^2) = o(1)$$

and

$$\int_{\Omega_1} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} + \int_{\Omega_1} |u_n|^2 = 1.$$

Set $\tilde{u}_n(x) = u_n(x - x^1)$. Then for any $R > 0$, since $\text{dist}(x^1, \partial\Omega_1) = r \sin \frac{\pi}{k} \geq Ck^{\frac{m}{N+2s-m}} \rightarrow +\infty$, $B_R(x^1) \subset \Omega_1$, i.e.

$$\int_{B_R(x^1)} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} + \int_{B_R(x^1)} |u_n|^2 \leq 1.$$

One can obtain

$$\int_{B_R(0)} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^2}{|x - y|^{N+2s}} + \int_{B_R(0)} |\tilde{u}_n|^2 \leq 1.$$

So we suppose that there is a $u \in H^s(\mathbb{R}^N)$, such that as $n \rightarrow +\infty$,

$$\tilde{u}_n \rightharpoonup u, \quad \text{in } H_{loc}^s(\mathbb{R}^N)$$

and

$$\tilde{u}_n \rightarrow u, \quad \text{in } L_{loc}^2(\mathbb{R}^N).$$

Since \tilde{u}_n is even in $x_j, j = 2, \dots, N$, it is easy to see that u is even in $x_j, j = 2, \dots, N$. On the other hand, from

$$\int_{\mathbb{R}^N} U_{x_1}^{p-1} Z_1 u_n = 0,$$

we get

$$\int_{\mathbb{R}^N} U^{p-1} \frac{\partial U}{\partial x_1} \tilde{u}_n = 0.$$

So, u satisfies

$$\int_{\mathbb{R}^N} U^{p-1} \frac{\partial U}{\partial x_1} u = 0. \quad (4.3)$$

Now, we claim that u satisfies

$$(-\Delta)^s u + u - pU^{p-1}u = 0 \quad \text{in } \mathbb{R}^N. \quad (4.4)$$

Indeed, we set

$$\tilde{E} = \left\{ \varphi : \varphi \in H^s(\mathbb{R}^N), \int_{\mathbb{R}^N} U^{p-1} \frac{\partial U}{\partial x_1} \varphi = 0 \right\}.$$

For any $R > 0$, let $\varphi \in C_0^\infty(B_R(0)) \cap \tilde{E}$ be any function, satisfying that φ is even in $x_j, j = 2, \dots, N$. Then $\varphi_1(x) = \varphi(x - x^1) \in C_0^\infty(B_R(x^1))$. We may identify $\varphi_1(x)$ as elements in E by redefining the values outside Ω_1 with the symmetry. By using (4.2) and Lemma A.2, we find

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} + \int_{\mathbb{R}^N} (u\varphi - pU^{p-1}u\varphi) = 0. \quad (4.5)$$

On the other hand, since u is even in $x_j, j = 2, \dots, N$, (4.5) holds for any $\varphi \in C_0^\infty(B_R(0)) \cap \tilde{E}$. By the density of $C_0^\infty(\mathbb{R}^N)$ in $H^s(\mathbb{R}^N)$, it is easy to show that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} + \int_{\mathbb{R}^N} (u\varphi - pU^{p-1}u\varphi) = 0, \quad \forall \varphi \in \tilde{E}. \quad (4.6)$$

We know $\varphi = \frac{\partial U}{\partial x_1}$ is a solution of (4.6), thus (4.6) is true for any $\varphi \in H^s(\mathbb{R}^N)$. One see that $u = 0$ because u is even in $y_i, i = 2, \dots, N$ and (4.4). As a result,

$$\int_{B_R(x^1)} u_n^2 = o_n(1), \quad \forall R > 0,$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow +\infty$.

Now, using the Lemma A.2, we obtained that for any $1 < \eta \leq N + 2s$, there is a constant $C > 0$, such that

$$U_{r_k}(x) \leq \frac{C}{(1 + |x - x^1|)^{N+2s-\eta}}, \quad x \in \Omega_1.$$

Thus,

$$o_n(1) = \int_{\Omega_1 \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} + \int_{\Omega_1} (u_n^2 - pK(x)U_{r_k}^{p-1}u_n^2)$$

$$\begin{aligned}
&= \int_{\Omega_1} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} + \int_{\Omega_1} u_n^2 \\
&\quad + C \left(\int_{B_{\frac{1}{2}R}(x^1)} + \int_{\Omega_1 \setminus B_{\frac{1}{2}R}(x^1)} \frac{1}{(1 + |x - x^1|)^{N+2s-\eta}} u_n^2 \right) \\
&\geq \frac{1}{2} \left(\int_{\Omega_1} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} + \int_{\Omega_1} u_n^2 \right) + o_n(1) + o_R(1),
\end{aligned}$$

which is impossible for large R .

As a result, we get a contradiction. \square

Proposition 4.2. *There is an integer $k_0 > 0$, such that for each $k \geq k_0$, there is a C^1 map from S_k to \mathcal{H} : $\omega = \omega(r)$, $r = |x^1|$, satisfying $\omega \in E$, and*

$$J'(\omega) \Big|_E = 0.$$

Moreover, there exists a constant $C > 0$ independent of k such that

$$\|\omega\|_s \leq Ck^{\frac{1}{2}} \left(\left(\frac{k}{r} \right)^{\frac{N+2s}{2} + \tau} + \frac{1}{r^{\frac{m}{2} + \tau}} \right), \quad (4.7)$$

where $\tau > 0$ is a small constant.

Proof. We will use the contraction theorem to prove it. By the following Lemma 4.3, $l(\omega)$ is a bounded linear functional in E . We know by Reisz representation theorem that there is an $l_k \in E$, such that

$$l(\omega) = \langle l_k, \omega \rangle.$$

So, finding a critical point for $J(\omega)$ is equivalent to solving

$$l_k + L\omega + R'(\omega) = 0. \quad (4.8)$$

By Proposition 4.1, L is invertible. Thus, (4.8) is equivalent to

$$\omega = A(\omega) := -L^{-1}(l_k + R'(\omega)).$$

Set

$$\bar{S}_k := \left\{ \omega \in E : \|\omega\|_s \leq k^{\frac{1}{2}} \left(\frac{k}{r} \right)^{\frac{N+2s}{2} + \tau} + \frac{k^{\frac{1}{2}}}{r^{\frac{m}{2} + \tau}} \right\}.$$

We shall verify that A is a contraction mapping from \bar{S}_k to itself. In fact, on one hand, for any $\omega \in \bar{S}_k$, by Lemmas 4.3 and 3.1, we obtain

$$\begin{aligned}
\|A(\omega)\| &\leq C(\|l\| + \|R'(\omega)\|) \\
&\leq C\|l\| + C\|\omega\|_s^{\min\{p, 2\}} \\
&\leq C \left(k^{\frac{1}{2}} \left(\frac{k}{r} \right)^{\frac{N+2s}{2} + \tau} + \frac{k^{\frac{1}{2}}}{r^{\frac{m}{2} + \tau}} \right) + C \left(k^{\frac{1}{2}} \left(\frac{k}{r} \right)^{\frac{N+2s}{2} + \tau} + \frac{k^{\frac{1}{2}}}{r^{\frac{m}{2} + \tau}} \right)^{\min\{p, 2\}} \\
&\leq k^{\frac{1}{2}} \left(\frac{k}{r} \right)^{\frac{N+2s}{2} + \tau} + \frac{k^{\frac{1}{2}}}{r^{\frac{m}{2} + \tau}}.
\end{aligned}$$

On the other hand, for any $\omega_1, \omega_2 \in \bar{S}_k$,

$$\begin{aligned}
\|A(\omega_1) - A(\omega_2)\| &= \|L^{-1}R'(\omega_1) - L^{-1}R'(\omega_2)\| \\
&\leq C\|R'(\omega_1) - R'(\omega_2)\| \\
&\leq C\|R''(\theta\omega_1 + (1-\theta)\omega_2)\|\|\omega_1 - \omega_2\|_s \\
&\leq \begin{cases} C(\|\omega_1\|_s^{p-1} + \|\omega_2\|_s^{p-1})\|\omega_1 - \omega_2\|_s, & \text{if } 1 < p < 2 \\ C(\|\omega_1\|_s + \|\omega_2\|_s)\|\omega_1 - \omega_2\|_s, & \text{if } p \geq 2 \end{cases} \\
&\leq \frac{1}{2}\|\omega_1 - \omega_2\|_s.
\end{aligned}$$

Then the result follows from the contraction mapping theorem. The estimate (4.7) follows Lemma 4.3.

The claim that $\omega(r)$ is continuously differentiable in r can be verified by using the same argument employed to proof Lemma 4.4 in [11]. \square

Lemma 4.3. *There is a constant $C > 0$ and a small constant $\tau > 0$, which are independent of k , such that*

$$\|l_k\| \leq C\left(k^{\frac{1}{2}}\left(\frac{k}{r}\right)^{\frac{N+2s}{2}+\tau} + \frac{k^{\frac{1}{2}}}{r^{\frac{m}{2}+\tau}}\right),$$

provided $k \geq k_0$ for some integer $k_0 > 0$.

Proof.

$$\begin{aligned}
|l(\varphi)| &= \left| \int_{\mathbb{R}^N} \sum_{i=1}^k U_{x^i}^p \varphi - K(x) \left(\sum_{i=1}^k U_{x^i} \right)^p \varphi \right| \\
&\leq C \int_{\mathbb{R}^N} \left| \left(\sum_{i=1}^k U_{x^i} \right)^p - \sum_{i=1}^k U_{x^i}^p \right| |\varphi| + \int_{\mathbb{R}^N} \left| (K(x) - 1) \left(\sum_{i=1}^k U_{x^i} \right)^p \varphi \right|.
\end{aligned} \tag{4.9}$$

From $m > \frac{N+2s}{N+2s+1}$, we deduce that

$$\frac{N+2s-m}{m} < N+2s < \left((N+2s)(p-1) + \frac{N+2s}{2} - \frac{p}{p+1}N \right) \left(\frac{p}{p+1} - \frac{1}{2} \right)^{-1}.$$

Hence, we can choose $\sigma \in (0, \frac{N+2s}{2})$ such that

$$\frac{p+1}{p} \left((N+2s)(p-1) + \frac{N+2s}{2} - \sigma \right) > N, \quad \frac{p}{p+1} - \frac{1}{2} < \frac{m}{N+2s-m} \sigma. \tag{4.10}$$

Using the fact $U_{x^i} \leq U_{x^1}$, ($x \in \Omega_1$) and Lemma A.2, we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^N} \left| \left(\sum_{i=1}^k U_{x^i} \right)^p - \sum_{i=1}^k U_{x^i}^p \right| |\varphi| = k \int_{\Omega_1} \left| \left(\sum_{i=1}^k U_{x^i} \right)^p - \sum_{i=1}^k U_{x^i}^p \right| |\varphi| \\
& \leq Ck \int_{\Omega_1} U_{x^1}^{p-1} \sum_{i=2}^k U_{x^i} |\varphi| \\
& \leq Ck \left(\int_{\Omega_1} \left(U_{x^1}^{p-1} \sum_{i=2}^k U_{x^i} \right)^{\frac{p+1}{p}} \right)^{\frac{p}{p+1}} \left(\int_{\Omega_1} |\varphi|^{p+1} \right)^{\frac{1}{p+1}} \\
& \leq Ck \left(\int_{\Omega_1} \left(\frac{1}{(1 + |x - x^1|)^{(N+2s)(p-1) + \frac{N+2s}{2} - \sigma}} \left(\frac{k}{r} \right)^{\frac{N+2s}{2} + \sigma} \right)^{\frac{p+1}{p}} \right)^{\frac{p}{p+1}} \left(\int_{\Omega_1} |\varphi|^{p+1} \right)^{\frac{1}{p+1}} \\
& = Ck^{\frac{p}{p+1}} \left(\frac{k}{r} \right)^{\frac{N+2s}{2} + \sigma} \left(\int_{\Omega_1} \left(\frac{1}{(1 + |x - x^1|)^{(N+2s)(p-1) + \frac{N+2s}{2} - \sigma}} \right)^{\frac{p+1}{p}} \right)^{\frac{p}{p+1}} \|\varphi\|_s \\
& \leq Ck^{\frac{1}{2}} \left(\frac{k}{r} \right)^{\frac{N+2s}{2} + \tau} \|\varphi\|_s,
\end{aligned} \tag{4.11}$$

for some small constant $\tau > 0$, where we have used (4.10).

On the other hand, we have

$$\begin{aligned}
& \int_{\mathbb{R}^N} \left| (K(x) - 1) \sum_{i=1}^k U_{x^i}^p \varphi \right| = k \int_{\mathbb{R}^N} \left| (K(x) - 1) U_{x^1}^p \varphi \right| \\
& \leq Ck \left(\int_{\mathbb{R}^N} |K(x) - 1|^{\frac{p+1}{p}} U_{x^1}^{p+1} \right)^{\frac{p}{p+1}} \left(\int_{\mathbb{R}^N} |\varphi|^{p+1} \right)^{\frac{1}{p+1}} \\
& \leq Ck \|\varphi\|_s \left(\left(\int_{B_{\frac{r}{2}}(x^1)} |K(x) - 1|^{\frac{p+1}{p}} U_{x^1}^{p+1} \right)^{\frac{p}{p+1}} + C \int_{\mathbb{R}^N \setminus B_{\frac{r}{2}}(x^1)} U_{x^1}^{p+1} \right)^{\frac{p}{p+1}} \\
& \leq C \|\varphi\|_s \left(\frac{k}{r^m} + \frac{k}{r^{(N+2s)p - \frac{p}{p+1}N}} \right) \\
& \leq C \frac{k^{\frac{1}{2}}}{r^{\frac{m}{2} + \tau}} \|\varphi\|_s,
\end{aligned} \tag{4.12}$$

for some small constant $\tau > 0$, where the last inequality is due to the assumption $\frac{N+2s}{N+2s+1} < m < N + 2s$.

Inserting (4.11)–(4.12) into (4.9), we can complete the proof. \square

Now, we are ready to prove our main theorem. Let $\omega = \omega(r)$ be the map obtained in Proposition 4.2. Define

$$F(r) = I(U_r + \omega), \quad \forall r \in S_k.$$

It follows from Lemma 6.1 in [11] that if r is a critical point of $F(r)$, then $U_r + \omega$ is a solution of (1.1).

Proof of Theorem 1.4. It follows from Propositions 4.2 and A.3 that

$$\begin{aligned}
F(r) &= I(U_r) + O(\|l\| \|\omega\| + \|\omega\|^2) \\
&= k \left(A - \frac{B_0 k^{N+2s}}{r^{N+2s}} + \frac{B_1}{r^m} + O\left(\frac{1}{r^{m+\sigma}}\right) + O\left(\left(\frac{k}{r}\right)^{N+2s+2\tau} + \frac{1}{r^{m+2\tau}}\right) \right) \\
&= k \left(A - \frac{B_0 k^{N+2s}}{r^{N+2s}} + \frac{B_1}{r^m} + O\left(\frac{1}{r^{m+\sigma}}\right) \right).
\end{aligned} \tag{4.13}$$

Define

$$F_1(r) := -\frac{B_0 k^{N+2s}}{r^{N+2s}} + \frac{B_1}{r^m} + O\left(\frac{1}{r^{m+\sigma}}\right).$$

We consider the following maximization problem

$$\max_{r \in S_k} F_1(r).$$

Suppose that \hat{r} is a maximizer, we will prove that \hat{r} is an interior point of S_k .

We can check that the function

$$G(r) = -\frac{B_0 k^{N+2s}}{r^{N+2s}} + \frac{B_1}{r^m}$$

has a maximum point

$$\tilde{r} = \left(\frac{B_0(N+2s)}{B_1 m} \right)^{\frac{1}{N+2s-m}} k^{\frac{N+2s}{N+2s-m}}$$

and

$$\frac{B_0 k^{N+2s}}{\tilde{r}^{N+2s}} = \frac{B_1}{\tilde{r}^m} \frac{m}{N+2s}.$$

By direct computation, we deduce that

$$\begin{aligned}
F_1(\tilde{r}) &= -\frac{B_0 k^{N+2s}}{\tilde{r}^{N+2s}} + \frac{B_1}{\tilde{r}^m} + O\left(\frac{1}{\tilde{r}^{m+\sigma}}\right) \\
&= \frac{B_1}{\tilde{r}^m} \left(1 - \frac{m}{N+2s} \right) + O\left(\frac{1}{\tilde{r}^{m+\sigma}}\right) \\
&= \frac{B_1^{\frac{N+2s}{N+2s-m}}}{B_0^{\frac{m}{N+2s-m}}} \frac{1}{\left(\frac{N+2s}{m}\right)^{\frac{N+2s}{N+2s-m}}} \left(\frac{N+2s}{2m} - 1 \right) k^{-\frac{(N+2s)m}{N+2s-m}} + O(k^{-\frac{(N+2s)m}{N+2s-m}-\sigma}).
\end{aligned}$$

On the other hand, we find

$$\begin{aligned}
&F_1\left(\left(\frac{B_0(N+2s)}{B_1 m} - \alpha\right)^{\frac{1}{N+2s-m}} k^{\frac{N+2s}{N+2s-m}}\right) \\
&= -\frac{B_0 k^{N+2s}}{\left(\frac{N+2s}{m} \frac{B_0}{B_1} - \alpha\right)^{\frac{N+2s}{N+2s-m}} k^{\frac{(N+2s)^2}{N+2s-m}}} + \frac{B_1}{\left(\frac{N+2s}{m} \frac{B_0}{B_1} - \alpha\right)^{\frac{m}{N+2s-m}} k^{\frac{m(N+2s)}{N+2s-m}}} + O(k^{-\frac{(N+2s)m}{N+2s-m}-\sigma}) \\
&= \frac{B_1^{\frac{N+2s}{N+2s-m}}}{B_0^{\frac{m}{N+2s-m}}} \frac{1}{\left(\frac{N+2s}{m} - \alpha \frac{B_1}{B_0}\right)^{\frac{N+2s}{N+2s-m}}} \left(\frac{N+2s}{m} - \alpha \frac{B_1}{B_0} - 1 \right) k^{-\frac{(N+2s)m}{N+2s-m}} + O(k^{-\frac{(N+2s)m}{N+2s-m}-\sigma}) \\
&< F_1(\tilde{r})
\end{aligned}$$

and similarly

$$\begin{aligned}
& F_1 \left(\left(\frac{B_0(N+2s)}{B_1 m} + \alpha \right)^{\frac{1}{N+2s-m}} k^{\frac{N+2s}{N+2s-m}} \right) \\
&= \frac{B_1^{\frac{N+2s}{N+2s-m}}}{B_0^{\frac{m}{N+2s-m}}} \frac{1}{\left(\frac{N+2s}{m} + \alpha \frac{B_1}{B_0} \right)^{\frac{N+2s}{N+2s-m}}} \left(\frac{N+2s}{m} + \alpha \frac{B_1}{B_0} - 1 \right) k^{-\frac{(N+2s)m}{N+2s-m}} + O(k^{-\frac{(N+2s)m}{N+2s-2m}-\sigma}) \\
&< F_1(\tilde{r}),
\end{aligned}$$

where we have used the fact that the function $f(x) = x^{-\frac{N+2s}{N+2s-m}}(x-1)$ attains its maximum at $x_0 = \frac{N+2s}{m}$ if $x \in [\frac{N+2s}{m} - \alpha, \frac{N+2s}{m} + \alpha]$.

The above estimates implies that \hat{r} is indeed an interior point of S_k . Thus

$$u_{\hat{r}} = U_{\hat{r}} + \omega_{\hat{r}}$$

is a solution of (1.1).

At last, we claim that $u_{\hat{r}} > 0$. Indeed, since $\|\omega_{\hat{r}}\|_s \rightarrow 0$ as $k \rightarrow \infty$, noticing the fact (see [11] for example) that

$$\|u_{\hat{r}}\|_s^2 = \int_{\mathbb{R}_+^{N+1}} |\nabla \tilde{u}_{\hat{r}}|^2 y^{1-2s} dx dy + \int_{\mathbb{R}^N} u_{\hat{r}}^2 dx,$$

where $\tilde{u}_{\hat{r}}(x, y)$ is the s -harmonic extension of $u_{\hat{r}}$ satisfying $\tilde{u}_{\hat{r}}(x, 0) = u_{\hat{r}}(x)$, we can use the standard argument to verify that $(u_{\hat{r}})_- = 0$ and hence $u_{\hat{r}} \geq 0$. Since $u_{\hat{r}}$ solves (2.3), we conclude by using the strong maximum principle that $u_{\hat{r}} > 0$. \square

APPENDIX A. ENERGY EXPANSION

In this section, we will give the energy expansion for the approximate solutions. Recall

$$x^i = \left(r \cos \frac{2(i-1)\pi}{k}, r \sin \frac{2(i-1)\pi}{k}, 0 \right), \quad i = 1, \dots, k,$$

$$\Omega_i = \left\{ x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\langle \frac{x'}{|x'|}, \frac{x^i}{|x^i|} \right\rangle \geq \cos \frac{\pi}{k} \right\}, \quad i = 1, 2, \dots, k,$$

and

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} + \frac{1}{2} \int_{\mathbb{R}^N} u^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} K(x) |u|^{p+1}.$$

Firstly, we will introduce Lemma B.1 ([30]).

Lemma A.1. *For any constant $0 < \sigma \leq \min\{\alpha, \beta\}$, there is a constant $C > 0$, such that*

$$\frac{1}{(1 + |y - x^i|)^\alpha} \frac{1}{(1 + |y - x^j|)^\beta} \leq \frac{C}{|x^i - x^j|^\sigma} \left(\frac{1}{(1 + |y - x^i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1 + |y - x^j|)^{\alpha+\beta-\sigma}} \right),$$

where $\alpha, \beta \geq 1$ are two constants.

Then, we have the following basic estimate:

Lemma A.2. *For any $x \in \Omega_1$, and $\eta \in (1, N + 2s]$, there is a constant $C > 0$, such that*

$$\sum_{i=2}^k U_{x^i} \leq C \frac{1}{(1 + |x - x^1|)^{N+2s-\eta}} \frac{k^\eta}{|x_1|^\eta} \leq C \frac{k^\eta}{|x_1|^\eta}.$$

Proof. The proof of this lemma is similar to Lemma A.1 in [29], we sketch the proof below for the sake of completeness.

For any $x \in \Omega_1$, we have for $i \neq 1$,

$$|x - x^i| \geq |x - x^1|, \quad \forall x \in \Omega_1,$$

which gives $|x - x^i| \geq \frac{1}{2}|x^i - x^1|$ if $|x - x^1| \geq \frac{1}{2}|x^i - x^1|$. On the other hand, if $|x - x^1| \leq \frac{1}{2}|x^i - x^1|$, then

$$|x - x^i| \geq |x^i - x^1| - |x - x^1| \geq \frac{1}{2}|x^i - x^1|.$$

So, we find

$$|x - x^i| \geq \frac{1}{2}|x^i - x^1|, \quad \forall x \in \Omega_1.$$

Thus,

$$\begin{aligned} \sum_{i=2}^k U_{x^i} &= \sum_{i=2}^k \frac{C}{1 + |x - x^i|^{N+2s}} = \sum_{i=2}^k \frac{C}{(1 + |x - x^i|)^{N+2s}} \\ &= C \sum_{i=2}^k \frac{1}{(1 + |x - x^i|)^\eta} \frac{1}{(1 + |x - x^i|)^{N+2s-\eta}} \\ &\leq C \frac{1}{(1 + |x - x^1|)^{N+2s-\eta}} \sum_{i=2}^k \frac{1}{|x^1 - x^i|^\eta}. \end{aligned}$$

Since

$$|x^i - x^1| = 2|x^1| \sin \frac{(i-1)\pi}{k}, \quad i = 2, \dots, k,$$

we have

$$\begin{aligned} \sum_{i=2}^k \frac{1}{|x^i - x^1|^\eta} &= \frac{1}{(2|x^1|)^\eta} \sum_{i=2}^k \frac{1}{(\sin \frac{(i-1)\pi}{k})^\eta} \\ &= \begin{cases} \frac{1}{(2|x^1|)^\eta} \sum_{i=2}^{\frac{k}{2}} \frac{1}{(\sin \frac{(i-1)\pi}{k})^\eta} + \frac{1}{(2|x^1|)^\eta}, & \text{if } k \text{ is even,} \\ \frac{1}{(2|x^1|)^\eta} \sum_{i=2}^{[\frac{k}{2}]} \frac{1}{(\sin \frac{(i-1)\pi}{k})^\eta}, & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

But

$$0 < c' \leq \frac{\sin \frac{(i-1)\pi}{k}}{\frac{(i-1)\pi}{k}} \leq C'', \quad j = 2, \dots, [\frac{k}{2}].$$

So, there is a constant $B > 0$, such that

$$\sum_{i=2}^k \frac{1}{|x^i - x^1|^\eta} = \frac{Bk^\eta}{|x^1|^\eta} + O\left(\frac{k}{|x^1|^\eta}\right).$$

Thus, we obtain

$$\sum_{i=2}^k U_{x^i} \leq C \frac{1}{(1 + |x - x^1|)^{N+2s-\eta}} \frac{k^\eta}{|x^1|^\eta} \leq C \frac{k^\eta}{|x^1|^\eta}.$$

□

Proposition A.3. *There is a small constant $\tau > 0$, such that*

$$\begin{aligned} I(U_r) &= k \left(A - \frac{1}{2} \sum_{j=2}^k \frac{\tilde{B}_0}{|x^1 - x^j|^{N+2s}} + \frac{B_1}{r^m} + O\left(\frac{1}{r^{m+\tau}} + \left(\frac{k}{r}\right)^{N+2s+\tau} \right. \right. \\ &\quad \left. \left. + \sum_{j=2}^k \frac{B_0}{|x^1 - x^j|^{N+2s+\tau}} \right) \right) \\ &= k \left(A - \frac{B_0 k^{N+2s}}{r^{N+2s}} + \frac{B_1}{r^m} + O\left(\frac{1}{r^{m+\tau}}\right) \right), \end{aligned}$$

where $A = (\frac{1}{2} - \frac{1}{p}) \int_{\mathbb{R}^N} U^p$, and $B_0, B_1 > 0$ are positive constants.

Proof. Using the symmetry,

$$\begin{aligned} \langle U_r, U_r \rangle_s + \langle U_r, U_r \rangle_{L^2(\mathbb{R}^N)} &= \sum_{i=1}^k \sum_{j=1}^k \int_{\mathbb{R}^N} U_{x^j}^p U_{x^i} \\ &= k \left(\int_{\mathbb{R}^N} U_{x^1}^{p+1} + \sum_{j=2}^k \int_{\mathbb{R}^N} U_{x^1}^p U_{x^j} \right) \quad (\text{A.1}) \\ &= k \int_{\mathbb{R}^N} U^{p+1} + k \sum_{i=2}^k \int_{\mathbb{R}^N} U_{x^1}^p U_{x^i}. \end{aligned}$$

It follows from Lemma A.1 that

$$\begin{aligned} \sum_{i=2}^k \int_{\mathbb{R}^N} U_{x^1}^p U_{x^i} &= C \int_{\mathbb{R}^N} \left(\frac{1}{1 + |x - x^1|^{N+2s}} \right)^p \sum_{i=2}^k \frac{1}{1 + |x - x^i|^{N+2s}} \\ &\leq C \sum_{i=2}^k \frac{1}{|x^1 - x^i|^{N+2s}} \int_{\mathbb{R}^N} \frac{1}{(1 + |x - x^1|)^{(N+2s)p}} + O\left(\sum_{i=2}^k \frac{1}{|x^1 - x^i|^{N+2s+\tau}}\right) \\ &= \sum_{i=2}^k \frac{C_1}{|x^1 - x^i|^{N+2s}} + O\left(\sum_{i=2}^k \frac{1}{|x^1 - x^i|^{N+2s+\tau}}\right). \end{aligned}$$

However,

$$\begin{aligned}
\sum_{i=2}^k \int_{\mathbb{R}^N} U_{x^1}^p U_{x^i} &= \int_{\mathbb{R}^N} \left(\frac{1}{1 + |x - x^1|^{N+2s}} \right)^p \sum_{i=2}^k \frac{1}{1 + |x - x^i|^{N+2s}} \\
&\geq \sum_{i=2}^k \int_{B_{\frac{|x^1 - x^i|}{2}}(x^1)} \left(\frac{1}{1 + |x - x^1|^{N+2s}} \right)^p \frac{1}{1 + |x - x^i|^{N+2s}} \\
&\quad + \sum_{i=2}^k \int_{B_{\frac{|x^1 - x^i|}{2}}(x^i)} \left(\frac{1}{1 + |x - x^1|^{N+2s}} \right)^p \frac{1}{1 + |x - x^i|^{N+2s}} \\
&\geq \sum_{i=2}^k \frac{C_2}{|x^1 - x^i|^{N+2s}} + O\left(\sum_{i=2}^k \frac{1}{|x^1 - x^i|^{N+2s+\tau}} \right).
\end{aligned}$$

Hence, there exists B'_0 (which maybe depend on k) in $[C_2, C_1]$, where C_1 and C_2 are independent of k , such that

$$\sum_{i=2}^k \int_{\mathbb{R}^N} U_{x^1}^p U_{x^i} = \sum_{i=2}^k \frac{B'_0}{|x^1 - x^i|^{N+2s}} + O\left(\sum_{i=2}^k \frac{1}{|x^1 - x^i|^{N+2s+\tau}} \right). \quad (\text{A.2})$$

Now, by symmetry, we see

$$\begin{aligned}
\int_{\mathbb{R}^N} K(x) U_r^{p+1} &= k \int_{\Omega_1} K(x) U_{x^1}^{p+1} + k(p+1) \int_{\Omega_1} K(x) \sum_{i=2}^k U_{x^1}^p U_{x^i} \\
&\quad + k \begin{cases} O\left(\int_{\Omega_1} U_{x^1}^{\frac{p+1}{2}} \left(\sum_{i=2}^k U_{x^i} \right)^{\frac{p+1}{2}} \right), & \text{if } 1 < p < 2, \\ O\left(\int_{\Omega_1} U_{x^1}^{p-1} \left(\sum_{i=2}^k U_{x^i} \right)^2 \right), & \text{if } p \geq 2. \end{cases}
\end{aligned}$$

For $x \in \Omega_1$, we have $|x - x^i| \geq \frac{1}{2}|x^i - x^1|$. By Lemma A.1,

$$\begin{aligned}
\sum_{i=2}^k U_{x^i} &\leq C \sum_{i=2}^k \frac{1}{(1 + |x - x^1|)^{N+2s-\kappa}} \frac{1}{(1 + |x - x^i|)^\kappa} \\
&\leq C \sum_{i=2}^k \frac{1}{|x^1 - x^i|^{N+2s-\kappa}} \left(\frac{1}{(1 + |x - x^1|)^\kappa} + \frac{1}{(1 + |x - x^i|)^\kappa} \right) \\
&\leq C \sum_{i=2}^k \frac{1}{|x^1 - x^i|^{N+2s-\kappa}} \frac{1}{(1 + |x - x^1|)^\kappa},
\end{aligned}$$

where $\kappa > 0$ satisfies $\min\{\frac{p+1}{2}(N+2s-\kappa), 2(N+2s-\kappa)\} > N+2s$. Hence, we get

$$\begin{aligned}
& \int_{\Omega_1} U_{x^1}^{\frac{p+1}{2}} \left(\sum_{i=2}^k U_{x^i} \right)^{\frac{p+1}{2}} \\
& \leq C \int_{\Omega_1} \frac{1}{(1+|x-x^1|)^{\frac{(N+2s)(p+1)}{2}}} \left(\sum_{i=2}^k \frac{1}{|x^i-x^1|^{N+2s-\kappa}} \right)^{\frac{p+1}{2}} \frac{1}{(1+|x-x^1|)^{\frac{p+1}{2}\kappa}} \\
& = C \left(\sum_{i=2}^k \frac{1}{|x^i-x^1|^{N+2s-\kappa}} \right)^{\frac{p+1}{2}} \int_{\Omega_1} \frac{1}{(1+|x-x^1|)^{\frac{(N+2s)(p+1)}{2}+\kappa}} \\
& \leq C \left(\sum_{i=2}^k \frac{1}{|x^i-x^1|^{N+2s-\kappa}} \right)^{\frac{p+1}{2}} \leq C \left(\frac{k}{r} \right)^{N+2s+\tau}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega_1} U_{x^1}^{p-1} \left(\sum_{i=2}^k U_{x^i} \right)^2 \\
& \leq C \int_{\Omega_1} \frac{1}{(1+|x-x^1|)^{(N+2s)(p-1)}} \left(\sum_{i=2}^k \frac{1}{|x^i-x^1|^{N+2s-\kappa}} \right)^2 \frac{1}{(1+|x-x^1|)^{2\kappa}} \\
& = C \left(\sum_{i=2}^k \frac{1}{|x^i-x^1|^{N+2s-\kappa}} \right)^2 \int_{\Omega_1} \frac{1}{(1+|x-x^1|)^{(N+2s)(p-1)+2\kappa}} \\
& = C \left(\sum_{i=2}^k \frac{1}{|x^i-x^1|^{N+2s-\kappa}} \right)^2 \leq C \left(\frac{k}{r} \right)^{N+2s+\tau}.
\end{aligned}$$

On the other hand,

$$\int_{\Omega_1} K(x) U_{x^1}^p \sum_{i=2}^k U_{x^i} = \int_{\Omega_1} U_{x^1}^p \sum_{i=2}^k U_{x^i} + \int_{\Omega_1} (K(x) - 1) U_{x^1}^p \sum_{i=2}^k U_{x^i}.$$

But, from Lemma A.1 and (A.2),

$$\begin{aligned}
& \int_{\Omega_1} U_{x^1}^p \sum_{i=2}^k U_{x^i} = \int_{\mathbb{R}^N} U_{x^1}^p \sum_{i=2}^k U_{x^i} - \int_{\mathbb{R}^N \setminus \Omega_1} U_{x^1}^p \sum_{i=2}^k U_{x^i} \\
& = \int_{\mathbb{R}^N} U_{x^1}^p \sum_{i=2}^k U_{x^i} + O\left(\left(\frac{k}{r}\right)^\sigma \int_{\mathbb{R}^N \setminus \Omega_1} U_{x^1}^{p-\sigma} \sum_{i=2}^k U_{x^i}\right) \\
& = \int_{\mathbb{R}^N} U_{x^1}^p \sum_{i=2}^k U_{x^i} + O\left(\left(\frac{k}{r}\right)^\sigma \sum_{i=2}^k \frac{1}{|x^i-x^1|^{N+2s}} \int_{\mathbb{R}^N \setminus \Omega_1} \left(U_{x^1}^{p-\sigma} + U_{x^i}^{p-\sigma} \right)\right)
\end{aligned}$$

$$= \sum_{i=2}^k \frac{B'_0}{|x^1 - x^i|^{N+2s}} + O\left(\left(\frac{k}{r}\right)^{N+2s+\tau}\right),$$

where $\sigma > 0$ satisfies $p - \sigma > 1$.

Moreover, similarly,

$$\begin{aligned} & \int_{\Omega_1} |K(x) - 1| U_{x^1}^p \sum_{i=2}^k U_{x^i} \\ &= \int_{\mathbb{R}^N} |K(x) - 1| U_{x^1}^p \sum_{i=2}^k U_{x^i} - \int_{\mathbb{R}^N \setminus \Omega_1} |K(x) - 1| U_{x^1}^p \sum_{i=2}^k U_{x^i} \\ &\leq \int_{B_{\frac{r}{2}}(x^1)} |K(x) - 1| U_{x^1}^p \sum_{i=2}^k U_{x^i} + C \int_{\mathbb{R}^N \setminus B_{\frac{r}{2}}(x^1)} U_{x^1}^p \sum_{i=2}^k U_{x^i} + O\left(\left(\frac{k}{r}\right)^{N+2s+\tau}\right) \\ &\leq \frac{C}{r^m} \int_{\mathbb{R}^N} U_{x^1}^p \sum_{i=2}^k U_{x^i} + O\left(\frac{1}{r^\sigma} \sum_{i=2}^k \frac{1}{|x^i - x^1|^{N+2s}} \int_{\mathbb{R}^N \setminus B_{\frac{r}{2}}(x^1)} \left(U_{x^1}^{p-\sigma} + U_{x^i}^{p-\sigma}\right)\right) \\ &\quad + O\left(\left(\frac{k}{r}\right)^{N+2s+\tau}\right) \\ &= O\left(\left(\frac{k}{r}\right)^{N+2s+\tau} + \frac{1}{r^{m+\tau}}\right). \end{aligned}$$

Hence,

$$\int_{\Omega_1} K(x) U_{x^1}^p \sum_{i=2}^k U_{x^i} = \sum_{j=2}^k \frac{B'_0}{|x^1 - x^j|^{N+2s}} + O\left(\left(\frac{k}{r}\right)^{N+2s+\tau} + \frac{1}{r^{m+\tau}}\right).$$

Finally,

$$\begin{aligned} & \int_{\Omega_1} K(x) U_{x^1}^{p+1} \\ &= \int_{\mathbb{R}^N} K(x) U_{x^1}^{p+1} - \int_{\mathbb{R}^N \setminus B_{\frac{2\pi r}{k}}(x^1)} K(x) U_{x^1}^{p+1} + \int_{\Omega_1 \setminus B_{\frac{2\pi r}{k}}(x^1)} K(x) U_{x^1}^{p+1} \\ &= \int_{\mathbb{R}^N} K(x) U_{x^1}^{p+1} + O\left(\int_{\mathbb{R}^N \setminus B_{\frac{2\pi r}{k}}(x^1)} K(x) U_{x^1}^{p+1}\right) \\ &= \int_{B_{\frac{r}{2}}(x^1)} K(x) U_{x^1}^{p+1} + \int_{\mathbb{R}^N \setminus B_{\frac{r}{2}}(x^1)} K(x) U_{x^1}^{p+1} + O\left(\int_{\mathbb{R}^N \setminus B_{\frac{2\pi r}{k}}(x^1)} K(x) U_{x^1}^{p+1}\right) \\ &= \int_{\mathbb{R}^N} U^{p+1} - \frac{B'_1}{r^m} + O\left(\frac{1}{r^{m+\tau}}\right) + O\left(\left(\frac{k}{r}\right)^{(N+2s)(p+1)-N}\right) \\ &= \int_{\mathbb{R}^N} U^{p+1} - \frac{B'_1}{r^m} + O\left(\frac{1}{r^{m+\tau}} + \left(\frac{k}{r}\right)^{N+2s+\tau}\right) \end{aligned}$$

since $(N + 2s)(p + 1) - N > N + 2s$.

So, we have proved

$$\int_{\mathbb{R}^N} K(x) U_r^{p+1} = k \left(\int_{\mathbb{R}^N} U^{p+1} + \sum_{j=2}^k \frac{B'_0}{|x^1 - x^j|^{N+2s}} - \frac{B'_1}{r^m} + O\left(\frac{1}{r^{m+\tau}} + \left(\frac{k}{r}\right)^{N+2s+\tau}\right) \right). \quad (\text{A.3})$$

Now, inserting (A.1)–(A.3) into $I(U_r)$, we complete the proof. \square

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